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DATA ASSIMILATION BY REPEATED CORRECTION OF MODEL FIELDS - THEORETICAL CONSIDERATIONS

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1. Introduction

The purpose of this note is to give a short account of a simple theory concerning the method of assimilating data during the FGGE* data assimilation experiment, and which will form the basis for a future operational data assimilation scheme. The method involves the continuous updating of model fields with observations (or values interpolated therefrom) as it is integrated forward in time. The theory presented follows closely those of Hoke and Anthes (1976) and Davies and Turner (1977), but whereas these assume a continuous spectrum of observation in time, this assumption will be relaxed somewhat to be more in keeping with the practical assimilation problem.

These two papers will henceforth be referred to on HA and DT respectively.

* FGGE refers to the First GARP Global Experiment.

2. The Assimilation Process

Following HA and DT, the linearised shallow water equations are considered for small amplitude disturbances to an atmosphere at rest on an f-plane. The equations are, in vorticity and divergence form:

$$\frac{\partial \zeta}{\partial t} + fD = 0 \tag{2.1}$$

$$\frac{\partial D}{\partial t} - f\zeta + g\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = 0 \tag{2.2}$$

$$\frac{\partial h}{\partial t} + HD = 0 \tag{2.3}$$

where ζ and D are the vorticity and divergence, H the scale height, h the height perturbation, and the remaining symbols have their normal meaning. If the symbol '$\wedge$' denotes the true atmospheric state, then the assimilation process is defined by

$$\frac{\partial \zeta}{\partial t} + fD = K_1(\hat{\zeta} - \zeta) \tag{2.4}$$

$$\frac{\partial D}{\partial t} - f\zeta + g\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = K_2(\hat{D} - D) \tag{2.5}$$

$$\frac{\partial h}{\partial t} + HD = K_3(\hat{h} - h) \tag{2.6}$$

Writing $z' = \hat{z} - z$ and making the assumption that the actual atmosphere $(\hat{\zeta}, \hat{D}, \hat{h})$ satisfies (2.1) to (2.3), equations (2.4) to (2.6) become

$$\frac{\partial \zeta'}{\partial t} + fD' + K_1\zeta' = 0 \tag{2.7}$$

$$\frac{\partial D'}{\partial t} - f\zeta' + g\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + K_2 D' = 0 \tag{2.8}$$

$$\frac{\partial h'}{\partial t} + HD' + K_3 h' = 0 \qquad (2.9)$$

If solution are assumed proportional to $e^{\lambda t + i(kx+ly)}$ then λ is given by the following equation

$$\begin{vmatrix} (\lambda+K_1) & f & 0 \\ -f & (\lambda+K_2) & -g(k^2+l^2) \\ 0 & H & (\lambda+K_3) \end{vmatrix} = 0 \qquad (2.10)$$

When $K_1 = K_2 = K_3 = K$, that is both height and wind fields are updated, the solutions are

$$\lambda_1 = -K \qquad (2.11)$$

$$\lambda_{2,3} = -K \pm i\sqrt{f^2+\omega^2} \qquad (2.12)$$

where $\omega^2 = gH(k^2 + l^2)$.

The first solution corresponds to a damped non-divergent and stationary 'meteorological' wave, whilst the other two correspond to damped gravity-interial waves. In each case the e-folding time of the adjustment process in K^{-1}.

Of more general interest are the cases when either wind or height are updated alone. In passing it can first be noted that updating the divergence field alone corresponds to the geostrophic adjustment theory of Temperton (1973). Both HA and DT consider the updating of the height field, but only the former treats correction of the wind field as well. Each publishes diagrams of the decay rate [real part of $-\lambda$ (DT)] or of the e-folding time [real part of $-\lambda^{-1}$ (HA)] of the adjustment process,

but each of these is, in some important respects, incorrect. Thus HA does not admit the existence of three real solutions to equation (2.10) whereas the precise shape of the curve in DT is incorrect (see Appendix). More seriously, the latter does not identify the branches of the curves correctly.

Solutions of equation (2.10) are given in figures 2.1 to 2.3 for various values of K_i. The e-folding time is plotted against the wavelength L, assuming a value of 5km for the scale height H. Also given is the natural frequency ω of the gravity waves. In all cases f is taken to be 10^{-4} S^{-1}.

In figure 2.1a $K_1 = K_2 = 0$ and $K_3 = 10^{-3}$ S^{-1}, and the curves should correspond closely with the 'h only' curves of HA. Unlike these, however, the existence of 3 real solutions is clearly shown. In addition, the gradients of the curves near the branch points are different to those implied by DT. (Note that the latter plot decay rate against frequency ω). Figure 2.1b depicts the wind update case where $K_1 = K_2 = 10^{-3}$ S^{-1} and $K_3 = 0$. Figures 2.2 and 2.3 show the corresponding curves for the non-zero values of K_i equal to 10^{-2} and 10^{-4} S^{-1} respectively.

The solution of (2.10) can be analysed analytically for various parameter ranges, and this is done in the Appendix. An important result of this analysis is that it is possible to identify the various branches of the curves with the 3 free modes of equations (2.1) to (2.3), and also to calculate their asymptotic values for small and large wavelengths. The identities of these branches are labelled in figure 2.1 and correspond to these of HA. They are at variance, however, with DT.

For large wavelengths the e-folding time of the gravity-inertial waves becomes large when updating height only, whereas for meteorological waves it is approximately K_3^{-1}. The reverse is true when updating wind only. At small wavelengths the e-folding time of the meteorological wave now becomes large when only updating height, with that of the gravity-inertial waves becoming approximately $2\ K_3^{-1}$. When updating wind only ($K_1 = K_2$) at small

wavelengths, the e-folding times of the meteorological and gravity-inertial modes are approximetely K_1^{-1} and $2K_1^{-1}$ respectively. These results are in agreement with the results of Temperton's (1973) theory of geostrophic adjustment. That is, at large wavelengths the meteorological mode responds more rapidly to changes in the mass field than in the wind field, and vice versa at short wavelengths.

A further important result of the analysis presented in the Appendix is that if the non-zero K_i is assumed to be much larger than f, the e-folding time of certain modes when only one variable is updated is proportional to K_i/ω^2. Thus for a given wavelength increasing K_i can be counterproductive. More specifically, when only heights are updated, the gravity wave modes are affected in this way, whereas, when only winds are updated, it is the meteorological mode. This behaviour is confirmed by figures 2.1 to 2.3 and suggests the following restriction on the value of K_i.

$$K_i = O(f) \qquad (2.13)$$

HA also considered the updating of vorticity alone. Although this would be difficult to achieve in practise the results are interesting and are shown in figure 2.4 for $K_1 = 10^{-3}\ S^{-1}$. The meteorological mode is damped with an e-folding time of approximately K_1^{-1} S, but the gravity - inertial waves are only damped slowly.

Note that at large wavelengths the meteorological wave is damped more rapidly by updating only the vorticity component of the wind. The physical reason for this advanced by HA is that adjustment takes place between the mass and wind fields through the divergence field. This is impeded at large wavelengths if the divergence field is also updated. At short wavelengths the magnitude of the divergence generated by the adjustment mechanism is large enough to overcome this constraint.

A general conclusion which can be drawn from the figures is that, for

a given wavelength, the greatest adjustment time of any of the modes is smallest when updating wind alone, and very much smaller at short wavelengths.

In general, any atmospheric state will consist of large amplitude meteorological waves together with small amplitude gravity waves. The prime purpose of data assimilation is to modify the meteorological modes of the model to correspond to the true atmospheric state. At the same time it is desirable to suppress the generation of spurious gravity wave modes. However, the latter can be controlled by various damping mechanisms though care should be exercised so that the adjustment of the meteorological modes is not impaired. In other words, the solutions of (2.10) which are of most importance to the assimilation problem, are those which can be identified with the free meteorological mode of (2.1) to (2.3). Reference to figure 2.1 shows that because of this it is essential to have both wind and height data, the former for updating short wavelength modes, and the latter for updating at large wavelengths.

3. Error Growth

In section 2 it was assumed that the atmosphere satisfied the same equations as the model, that the observations were perfect, and that they were defined at all points in space and time. This is, of course, far from the truth in a practical assimilation scheme, and some attempt at assessing the impact of these errors is made in this section.

The error in the assumption that the atmosphere satisfies the same equation as the model is reflected in the growth of model error as it is integrated forward in time. Thus a minimum requirement is that the rate of adjustment of the model to the observed fields should be greater than the rate of growth of model error. This is not particularly helpful in specifying a limit on the values of K_i since most published values are concerned with average growth rates over well defined periods such as 6 or 12 hours. e.g.

Baumhefner and Downey (1978). During assimilation it is to be expected that the error growth rate willbe varying continuously.

Some assessment of the impact of observational errors can be made by replacing $\hat{\zeta}$ etc in (2.4) to (2.6) with $\zeta_{obs} = \hat{\zeta} - \epsilon_1^o$ where ζ_{obs} is the observed value of the vorticity and ϵ_1^o is its error. Thus (2.4) to (2.6) become

$$\frac{\partial \zeta'}{\partial t} + fD' + K_1 \zeta' = K_1 \epsilon_1^o \qquad (3.1)$$

$$\frac{\partial D'}{\partial t} - f\zeta' + K_2 D' + g\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = K_2 \epsilon_2^o \qquad (3.2)$$

$$\frac{\partial h'}{\partial t} + HD' + K_3 h' = K_3 \epsilon_3^o \qquad (3.3)$$

Following DT the error fields are decomposed into normal modes of the homogeneous equations and, because of linearity, each of these modes can be considered separately.

Writing

$$\epsilon_i^o = A_i e^{i\sigma t + i(kx+ly)} \qquad (3.4)$$

and defining the error response as

$$B_i e^{i\sigma t + i(kx+ly)} \qquad (3.5)$$

the equations for the amplitude of the error response are, from (3.1) to (3.3):

$$\begin{pmatrix} (i\sigma + K_1) & f & 0 \\ -f & (i\sigma + K_2) & -\omega^2/H \\ 0 & H & (i\sigma + K_3) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} K_1 A_1 \\ K_2 A_2 \\ K_3 A_3 \end{pmatrix} \qquad (3.6)$$

If only the case $K_1 = K_2$ is considered, i.e. wind update using both vorticity and divergence components equally, then the solution for the amplitudes of the error response can be written as

$$\left[\gamma_1\omega^2+\gamma_3\left(\gamma_1^2+f^2\right)\right]\begin{pmatrix}B_1\\B_2\\B_3\end{pmatrix}=\begin{pmatrix}\left(\omega^2+\gamma_1\gamma_3\right) & -f\gamma_3 & -f\omega^2/H\\ f\gamma_3 & \gamma_1\gamma_3 & \gamma_1\omega^2/H\\ -fH & -\gamma_1 H & \left(f^2+\gamma_1^2\right)\end{pmatrix}\begin{pmatrix}K_1A_1\\K_1A_2\\K_3A_3\end{pmatrix} \tag{3.7}$$

where $\gamma_i = i\sigma + K_i$

DT considered only the case when $K_1 = K_2 = 0$. In this

$$\frac{B_1}{A_3} = -\frac{f\omega^2}{H}\cdot\frac{K_3}{K_3\left(f^2-\sigma^2\right)+i\sigma\left(\omega^2+f^2-\sigma^2\right)} \tag{3.8}$$

$$\frac{B_3}{A_3} = \frac{K_3\left(f^2-\sigma^2\right)}{K_3\left(f^2-\sigma^2\right)+i\sigma\left(\omega^2+f^2-\sigma^2\right)} \tag{3.9}$$

Thus

$$\left|\frac{B_3}{A_3}\right|^2 = \left\{1+\frac{\sigma^2\left(\omega^2+f^2-\sigma^2\right)^2}{K_3^2\left(f^2-\sigma^2\right)^2}\right\}^{-1} \tag{3.10}$$

This should correspond with DT's equation 27 but, because of a mistake in the latter, it does not. This is unimportant since the conclusions which are drawn remain valid. That is, high frequency error modes $(\sigma >> f)$ induce only a small amplitude response.

A more interesting question centres on the response to the resonant frequencies of the normal undamped modes. i.e.

$$\sigma_1 = 0 \tag{3.11}$$

$$\sigma_{2,3} = \pm \left(\omega^2 + f^2\right)^{1/2} \tag{3.12}$$

This is important in the assimilation schemes currently being studied including that used during FGGE. In these discrete observations are used over a period of time to correct the model as it is integrated forward.

If the correct atmospheric mode is

$$X_i\, e^{i\nu t + i(kx+ly)} \tag{3.13}$$

But is observed by $[Y_i]_{t=0}$ where

$$Y_i = X_i\, e^{i\nu t + i(kx+ly)} - \epsilon_i^0 e^{i\sigma t + i(k'x+l'y)} \tag{3.14}$$

then the actual error is

$$\mathcal{E}_i^0 = \epsilon_i^0 e^{i(k'x+l'y)} + X_i e^{i(kx+ly)}\left(e^{i\nu t} - 1\right) \tag{3.15}$$

This formulation is unhelpful in the present analysis since for these simple linearised equations the natural frequency of the meteorological mode ν is zero. However, the following analogous problem can be posed which corresponds to the more general case of $\nu \neq 0$

The atmospheric mode is now specified as

$$X_i\, e^{i(kx+ly)} \tag{3.16}$$

and is observed by

$$Y_i = X_i\, e^{i\nu t + i(kx+ly)} - \epsilon_i^0 e^{i(k'x+l'y)} \tag{3.17}$$

Hence

$$\mathcal{E}_i^o = \epsilon_i^o e^{i(k'x+l'y)} + X_i e^{i(kx+ly)}\left(1-e^{i\nu t}\right) \qquad (3.18)$$

$$\mathcal{E}_i^o \simeq \epsilon_i^o e^{i(k'x+l'y)} - X_i\, i\nu t\, e^{i(kx+ly)} \qquad (3.19)$$

where the assumption $\nu t \ll 1$ is a necessary condition to ensure small error levels.

Thus

$$\mathcal{E}_{ii}^o = \epsilon_i^o e^{i(k'x+l'y)} \qquad (3.20)$$

has zero frequency and is a resonant mode.

Equation (3.7) becomes, for $\sigma = 0$

$$\left[K_1\omega^2 + K_3\left(K_1^2+f^2\right)\right]\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \left(\omega^2+K_1K_3\right) & -fK_3 & -f\omega^2/H \\ fK_3 & K_1K_3 & K_1\omega^2/H \\ -fH & K_1H & \left(f^2+K_1^2\right) \end{pmatrix}\begin{pmatrix} K_1A_1 \\ K_1A_2 \\ K_3A_3 \end{pmatrix} \qquad (3.21)$$

The observational error spectrum will be dominated by shortwavelength noise and thus, assuming $\omega \gg f$ and $K_1 = K_3 = K = O(f)$, equation (3.21) becomes

$$\begin{pmatrix} B_1 \\ B_2 \\ \frac{f}{H}B_3 \end{pmatrix} = \begin{pmatrix} 1 & O(f^2/\omega^2) & -1 \\ O(f^2/\omega^2) & O(f^2/\omega^2) & K/f \\ O(f^2/\omega^2) & O(f^2/\omega^2) & O(f^2/\omega^2) \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ \frac{f}{H}A_3 \end{pmatrix} \qquad (3.22)$$

There is a direct response to the forcing in the wind field and the amplitude of the response is of the same order as that of the error field. The response in the height field is an order of magnitude smaller.

When either K_1 or K_3 is zero, equation (3.21) may be used directly without the scaling necessary to derive (3.22). Thus when $K_1 = 0$ (height update only)

$$\left(\beta_1 , \beta_2 , \frac{f}{H}\beta_3\right) = \left(-\omega^2/f , 0 , 1\right)\frac{f}{H}A_3 \qquad (3.23)$$

and when $K_3 = 0$ (wind update only)

$$\left(\beta_1 , \beta_2 , \frac{f}{H}\beta_3\right) = \left(A_1 , 0 , \frac{f}{\omega^2}\left(K_1 A_2 - f A_1\right)\right) \qquad (3.24)$$

From (3.22) to (3.24) an essential requirement is seen to be that the amplitudes A_i of the observational error should be as small as possible. In addition (3.24) highlights the danger of too large a value of K_1 forcing a large response in the height field from errors in the divergence field. For short wavelength noise it can be assumed that $A_2 = O(A_1)$ and so a realistic requirement on K_1 from (3.24) is that

$$K_1 \leq O(f) \qquad (3.25)$$

In a practical assimilation scheme some form of interpolation to model grid points is necessary.

If (x,y) is a point in the model atmosphere and $(x+x_i , y+y_i)$ are observational points, the contribution to the interpolation error from those of the observations is, assuming linear interpolation

$$\begin{aligned}\mathcal{E}_{1i} &= \sum_i A_i w_i e^{i\left[k'(x+x_i) + l'(y+y_i)\right]} \\ &= e^{i(k'x+l'y)} \sum_i A_i w_i e^{i(k'x_i + l'y_i)}\end{aligned} \qquad (3.26)$$

where W_i are the interpolation weights.

Clearly a method which minimises the amplitude of this error would be desirable, and this immediately suggests the use of optimum interpolation (Gandin 1963, Rutherford 1972). However, this minimises the total mean square error including that due to a background field. Since the latter is typically a model prediction whose errors are likely to be dominated by the longer wavelength modes, the position is rather more complicated than the previous simple argument suggests.

There remains the contribution to the error forcing (3.19) due to using discrete observations to continuously update the model over a period of time. i.e.

$$\mathcal{E}^{o}_{2i} = - X_i \, i\nu t \, e^{i(kx+ly)} \tag{3.28}$$

Assuming an error response

$$\left(C_{oi} + C_{1i} t\right) e^{i(kx+ly)} \tag{3.28}$$

then it can be shown that

$$\propto \underline{C_1} = -i\nu \underline{\underline{M}} \cdot \underline{KX} \tag{3.29}$$

and

$$\propto \underline{C_o} = - \underline{\underline{M}} \cdot \underline{C_1} \tag{3.30}$$

The coefficient $\propto$ and matrix $\underline{\underline{M}}$ are the same as those which appear on the left and right hand sides respectively of (3.21), C_i^T the vector $(C_{i1}, C_{i2} C_{i3})$ and $(\underline{KX})^T$ the vector $(K_1 X_1, K_2 X_2, K_3 X_3)$. To ensure that the contribution to the error from $\underline{C_i}$ is no greater

than that due to the observational error, it can be deduced from (3.28) and (3.29) that

$$\nu t \leq O(\epsilon_{oi}/x_i) \tag{3.31}$$

and from (3.29) and (3.30) that

$$\nu/K_i \leq O(\epsilon_{oi}/x_i) \tag{3.32}$$

Using (3.31), (3.32) can be replaced by

$$K_i t \geq O(1) \tag{3.33}$$

4. Conclusions

It is convenient to define a period of assimilation T_a as the period of time over which observations, valid at a single time, are used to update the model, and a typical synoptic time scale T_s for the meteorological fields. The maximum difference between the validity time of an observation and the current model time will be the same order as T_a.

In section 3 it was stated that a minimum requirement was that the rate of adjustment of the model to the observations (i.e. the rate of data assimilation) should be greater than the rate of growth of model error. This cannot be translated into an explicit statement concerning the lower bound of the parameters K_i, but should always be borne in mind.

If it is assumed that $\epsilon_{oi}/x_i \ll 1$ then (3.31) implies

$$T_a \ll T_s \tag{4.1}$$

and (3.33)

$$K_i T_a \geq O(1) \tag{4.2}$$

These are both consequences of using data valid only at discrete points in time. Inequality (4.2) is well satisfied if

$$K_i T_a \gg 1 \tag{4.3}$$

and this also ensures that, if both height and winds are updated, the assimilation will converge to the observed state within the period T_a. This is important when observations do not occur at the same points in successive assimilation periods. Such data would not be fully utilised if K_i were less than that implied by (4.3).

An upper bound is imposed on K_i when the effects of observational error and the updating of only one of the variables (height or wind) is considered. These bounds are given in (3.25) and (2.13) respectively and together give rise to the following restriction

$$K_i \leq O(f) \tag{4.4}$$

Finally it was demontrated that using discrete observations in time gives rise to a resonant error mode forced by the observational error. This effect can be lessened through reducing the observational error by using a suitable interpolation scheme. Optimum interpolation would seem to be the most suitable choice, particularly because in its multivariate form it would mitigate the effects of observing only one variable. A correction to more than one model variable could be made through the use of multivariate statistics.

An unfortunate consequence of (4.4) is that, since $f^{-1} = O(T_s)$,

$$K_i^{-1} \geq O(T_s) \tag{4.5}$$

and hence (4.1) implies

$$K_i T_a \ll 1 \tag{4.6}$$

which is in direct contrast to (4.3). Clearly in a practical assimilation scheme a compromise is necessary and a straightforward

suggestion is to put

$$K_i T_a = O(1) \qquad (4.7)$$

The use of some form of interpolation is now essential. Interpolation can produce a more uniform spread of data with a smaller error level. Thus the relaxation of (4.3) is compensated by interpolated observations producing a more uniform spectrum of data in time. The greater coverage of data for each variable together with its lower error level also enables (4.6) to relaxed.

Since T_s is typically of the order of one day, (4.1) suggests a range for T_a of 1 to 3 hours. Then (4.7) implies $K_i \simeq 10^{-3}$ to 10^{-5} s^{-1}. Lyne (1978) demonstrated that the parameter λ, which scales the corrections to the model fields at each time step, in the FGGE scheme, is related to K_i through

$$\lambda = \frac{2K\Delta t}{1+2K\Delta t} \qquad (4.8)$$

Therefore with Δt = 450 secs in the FGGE scheme $K = 10^{-3}$ s^{-1} corresponds to λ = 0.55, and $K = 10^{-5}$ s^{-1} corresponds to λ = 0.01. During FGGE λ was varied linearily between zero and 0.5 during a 3 hour assimilation period. These figures are consistent with the values indicated by this theory.

The theory presented in this note is only valid on the so-called f plane. The extension to an equatorial β plane is given by DT for two special cases: (a) updating the north-south velocity field only and (b) updating only the zonal velocity and mass fields. They found that, subject to certain restrictions, an adequate assimilation of the data could be achieved.

There is no doubt that, due to the decoupling of equations (2.7) to (2.9) when $f = 0$, that observations of the wind field are of

paramount importance in equatorial latitudes. Naively, f may be set equal to zero in (2.10) and the following solutions obtained:

$$\text{(i)} \qquad K_1 = K_2 = 0 \quad ; \quad \lambda_1 = 0$$

$$\lambda_{2,3} = -\frac{K_3}{2} \pm \left(K_3^2 - 4\omega^2\right)^{1/2} \qquad (4.8)$$

$$\text{(ii)} \qquad K_3 = 0, \quad K_1 = K_2 \; ; \; \lambda_1 = -K_1$$

$$\lambda_{2,3} = -\frac{K_1}{2} \pm \left(K_1^2 - 4\omega^2\right)^{1/2} \qquad (4.9)$$

Thus observations of the mass field appear irrelevant, at least on the f = 0 plane. The principal conclusions (3.31) and (3.33) of the error analysis remain unaltered.

Finally some mention should be made of the implications when finite differences are employed. The finite difference analogue of (2.7) to (2.9) is, when centred differences in time are used,

$$\left(S'_{n+1} - S'_{n-1}\right)/2\Delta t + fD'_n + K_1 S'_{n+1} = 0 \qquad (4.10)$$

$$\left(D'_{n+1} - D'_{n-1}\right)/2\Delta t - fS'_n + g\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + K_2 D'_{n+1} = 0 \qquad (4.11)$$

$$\left(h'_{n+1} - h'_{n-1}\right)/2\Delta t + HD'_n + K_3 h'_{n+1} = 0 \qquad (4.12)$$

where the subscript refers to the time level and Δt in the time step. Lyne (1978) has shown that this is equivalent to the method of data assimilation employed during FGGE.

Assuming solutions proportional to $\mu^{n\Delta t} e^{i(kx+ly)}$ where n is the number of time steps, it can be shown that when $K_1 = K_2 = K_3 = K$, the amplification factor μ is given by

$$\frac{(1+2K\Delta t)\mu^2 - 1}{2\mu\Delta t} = 0 \quad \text{or} \quad \pm i\left(f^2+\omega^2\right)^{1/2} \qquad (4.13)$$

Thus for the meteorological mode

$$\mu = (1+2K\Delta t)^{-1/2} \qquad (4.14)$$

and this should be compared with $e^{-K\Delta t}$ for the continuous case. Thus the rate of convergence to the observed state is only $e^{K\Delta t}(1+2K\Delta t)^{-1/2}$ as fast as implied by the continuous equations.

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Appendix

It is convenient to scale λ, K_i and ω in (2.10) with respect to f. i.e. $\lambda \to \lambda/f$, $K_i \to K_i/f$ and $\omega \to \omega/f$.

The equation then becomes

$$\lambda^3 + (K_1+K_2+K_3)\lambda^2 + (K_1K_2 + K_2K_3 + K_3K_1 + 1+\omega^2)\lambda + (\omega^2K_1 + K_3 + K_1K_2K_3) = 0 \qquad (A1)$$

Solutions to (A1) are sought of the form

$$\lambda = \mathcal{E}^p \sum_{n=0}^{\infty} q_n \mathcal{E}^{nq} \qquad (A2)$$

where $\mathcal{E}$ is some suitable small parameter.

Short wavelength case, $\omega >> 1$

Putting $\mathcal{E} = \omega^{-1}$, substituting (A2) into (A1) and equating like powers of $\mathcal{E}$, it can be shown that p = −1, q = 1 and hence

$$\lambda_1 = -K_1 + O(\omega^{-2}) \qquad (A3)$$

$$\lambda_{2,3} = \pm i\omega - \frac{(K_2+K_3)}{2} \pm \frac{i\omega^{-1}}{2}\left[1 - \tfrac{1}{4}(K_2-K_3)^2\right] + O(\omega^{-2}) \qquad (A4)$$

Then the adjustment of the meteorological mode is seriously affected if $K_1 = 0$ (no vorticity update). If $K_2 = K_3$ (no divergence or height update) then the adjustment of the gravity modes is impaired.

Long wavelength case, $\omega << 1$

With $\mathcal{E} = \omega$ it is found that p = 0, q = 2 and that the solutions to (A1) are

$$\lambda_1 = -K_3 + O(\omega^2) \qquad (A5)$$

$$\lambda_{2,3} = -\frac{(K_1+K_2)}{2} \pm i\left[1 - \tfrac{1}{4}(K_1-K_2)^2\right]^{1/2} + O(\omega^2) \qquad (A6)$$

In this case the adjustment of the meteorological mode is impaired if $K_3 = 0$ (no height update) and that of the now mainly inertial modes if

$K_1 = K_2 = 0$ (no wind update).

None of the above expansions is valid if $K_i = O(\varepsilon)$. To investigate the effects of extreme values of K_i only the two cases of most interest are considered. These are height update only ($K_1 = K_2 = 0$, $K_3 = K$) and wind update only ($K_1 = K_2 = K$, $K_3 = 0$). For height update only, (A1) becomes

$$\lambda^3 + K\lambda^2 + (1+\omega^2)\lambda + K = 0 \qquad \text{(A7)}$$

If $\lambda^*(K)$ is a solution to (A7) then the solutions to (A1) for the wind update only case can be shown to be

$$\lambda = \lambda^*(-K) - K \qquad \text{(A8)}$$

Small K case, $K \ll 1$

Putting $\varepsilon = K$, then p=0 and q=1 and the solutions for height update only are:

$$\lambda_1 = -K/(1+\omega^2) + O(K^3) \qquad \text{(A9)}$$

$$\lambda_{2,3} = \pm i(1+\omega^2)^{1/2} - \frac{K}{2} + \frac{K}{2(1+\omega^2)} + O(K^2) \qquad \text{(A10)}$$

Hence for wind update only

$$\lambda_1 = -K\omega^2/(1+\omega^2) + O(K^3) \qquad \text{(A11)}$$

$$\lambda_{2,3} = \pm i(1+\omega^2) - \frac{K}{2} - \frac{K}{2(1+\omega^2)} + O(K^2) \qquad \text{(A12)}$$

Large K case, $K \gg 1$

Putting $\varepsilon = K^{-1}$ it is found that $p = -1$ and $q = 2$ or that $p = 0$ and $q = 1$. The solutions are then

$$\lambda_1 = -K + \omega^2/K + O(K^{-3}) \qquad \text{(A13)}$$

$$\lambda_{2,3} = \pm i - \omega^2/2K + O(K^{-2}) \qquad \text{(A14)}$$

for height update only, and

$$\lambda_1 = -\omega^2/K + O(K^{-3}) \tag{A15}$$

$$\lambda_{2,3} = -K \pm i + \omega^2/2K + O(K^{-2}) \tag{A16}$$

for wind update only.

These results have important consequences for the range of suitable K values, and their implications are discussed in section 2.

Shape of curves for λ near branch points

In section 2 it was intimated that the shape of the curves for the solution of (A7), is shown incorrectly by DT. In particular, the gradients near the branch points are incorrect.

Let

$$f(\lambda,\omega) = \lambda^3 + K\lambda^2 + (1+\omega^2)\lambda + K \tag{A17}$$

$$g(\lambda,\omega) = \frac{\partial f}{\partial \lambda} = 3\lambda^2 + 2K\lambda + 1+\omega^2 \tag{A18}$$

then at the branch point ω^*, the corresponding values of $\lambda = \lambda^*$ satify

$$f(\lambda^*, \omega^*) = 0 \tag{A19}$$

$$g(\lambda^*, \omega^*) = 0 \tag{A20}$$

Choosing $|\delta| \ll 1$ and assuming that $\lambda^* + \epsilon$ is the solution of (A7) corresponding to $\omega^* + \delta$, then the gradient of the λ curve is

$$\lim_{\delta \to 0} \left[\frac{\epsilon}{\delta}\right]_{\text{real part}} \tag{A21}$$

Using (A19) and (A20) it can be shown that

$$3\lambda^*\epsilon^2 + \epsilon^3 + K\epsilon^2 - (2\omega^*\delta + \delta^2)(\lambda^* + \epsilon) = 0 \tag{A22}$$

and hence

$$\epsilon^2 = -\frac{2\omega^*\lambda^*\delta}{3\lambda^* + K} + O(\delta^2) \tag{A23}$$

Therefore

$$\lim_{\delta \to 0} \left[\frac{\epsilon}{\delta}\right]_{\text{real part}} \sim O\left(\delta^{-1/2}\right) \to \infty \quad ; \epsilon^2 > 0 \tag{A24}$$

$$\lim_{\delta \to 0} \left[\frac{\epsilon}{\delta}\right]_{\text{real part}} \sim O(1) \quad ; \epsilon^2 < 0 \tag{A25}$$

The extension both to the co-ordinates used in this note and to the case $K_1 = K_2 = K$, $K_3 = 0$ is trivial, and the result (A24) holds in both instances.

Figures

All figures are of e-folding time T against wavelength L assuming a scale height H of 5 km. Also given is the equivalent frequency ω.

Figure 2.1a	,	$K_1 = K_2 = 0$, $K_3 = 10^{-3}\ s^{-1}$	(height update only)
Figure 2.1b	,	$K_1 = K_2 = 10^{-3}\ s^{-1}$, $K_3 = 0$	(wind update only)
Figure 2.2a	,	$K_1 = K_2 = 0$, $K_3 = 10^{-2}\ s^{-1}$	(height update only)
Figure 2.2b	,	$K_1 = K_2 = 10^{-2}\ s^{-1}$, $K_3 = 0$	(wind update only)
Figure 2.3a	,	$K_1 = K_2 = 0$, $K_3 = 10^{-4}\ s^{-1}$	(height update only)
Figure 2.3b	,	$K_1 = K_2 = 10^{-4}\ s^{-1}$, $K_3 = 0$	(wind update only)
Figure 2.4	,	$K_1 = 10^{-3}\ s^{-1}$, $K_2 = K_3 = 0$	(vorticity update only)

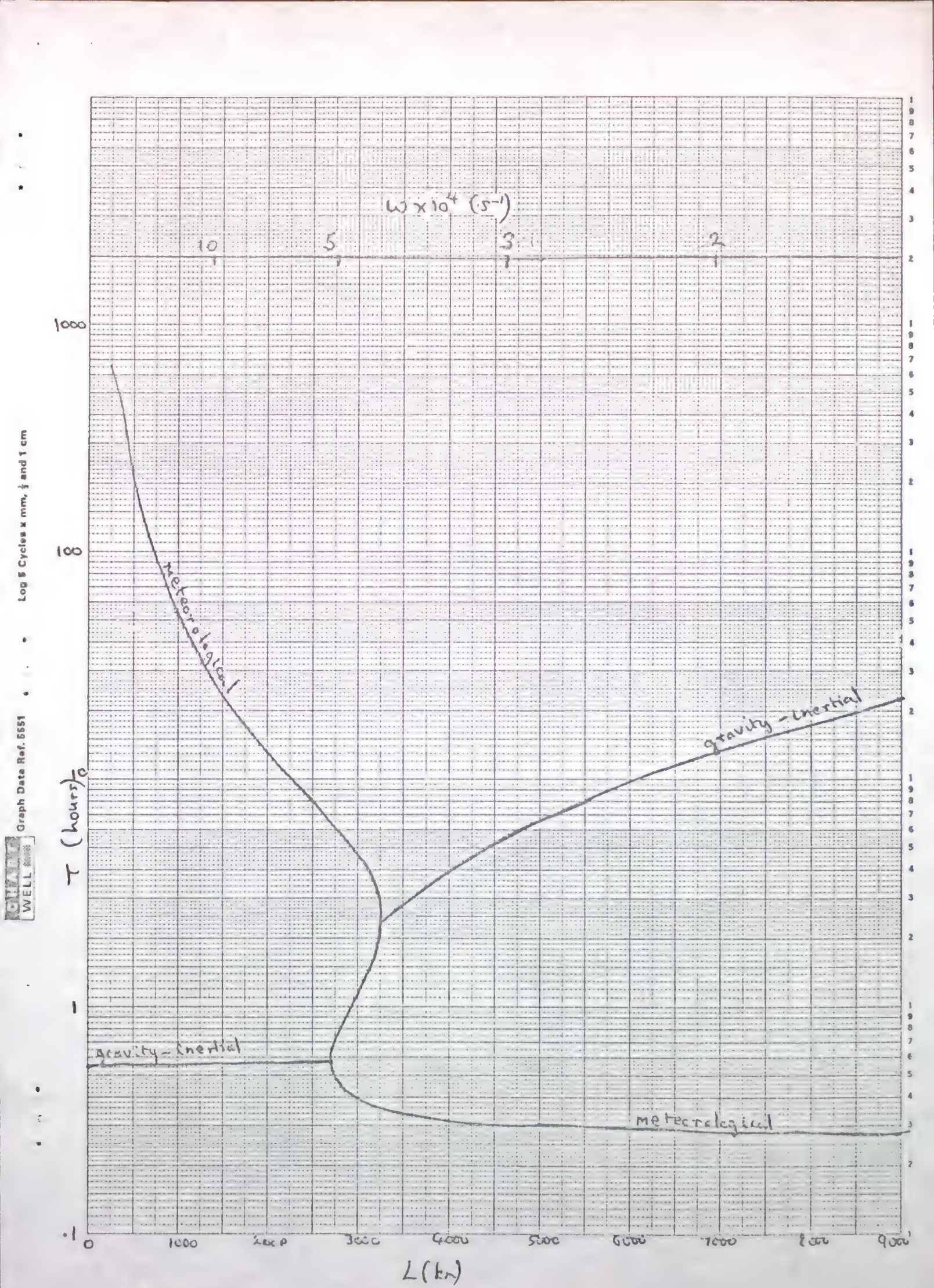

ω x 10⁴ (s⁻¹)
10
5
3
2
1000
100
1
·1
T (hours)
L (km)
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
meteorological
gravity - inertial
gravity - inertial
meteorological
Log 5 Cycles x mm, ½ and 1 cm
Graph Data Ref. 5551
WELL

Figure 2-1a

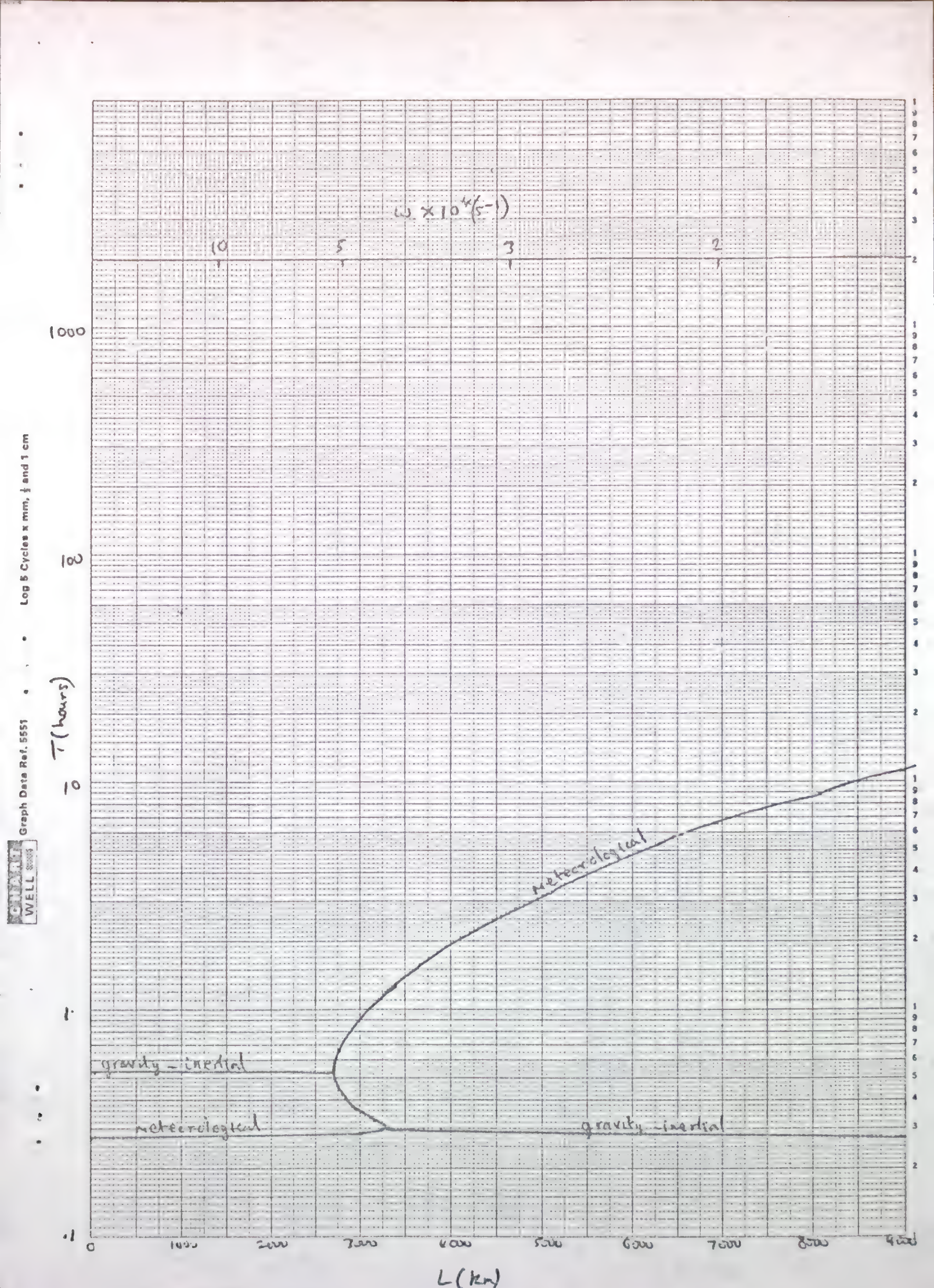

ω × 10⁴ (s⁻¹)
10
5
3
2
1000
100
10
1
.1
T (hours)
meteorological
gravity - inertial
meteorological
gravity - inertial
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
L (km)

Figure 2.1b

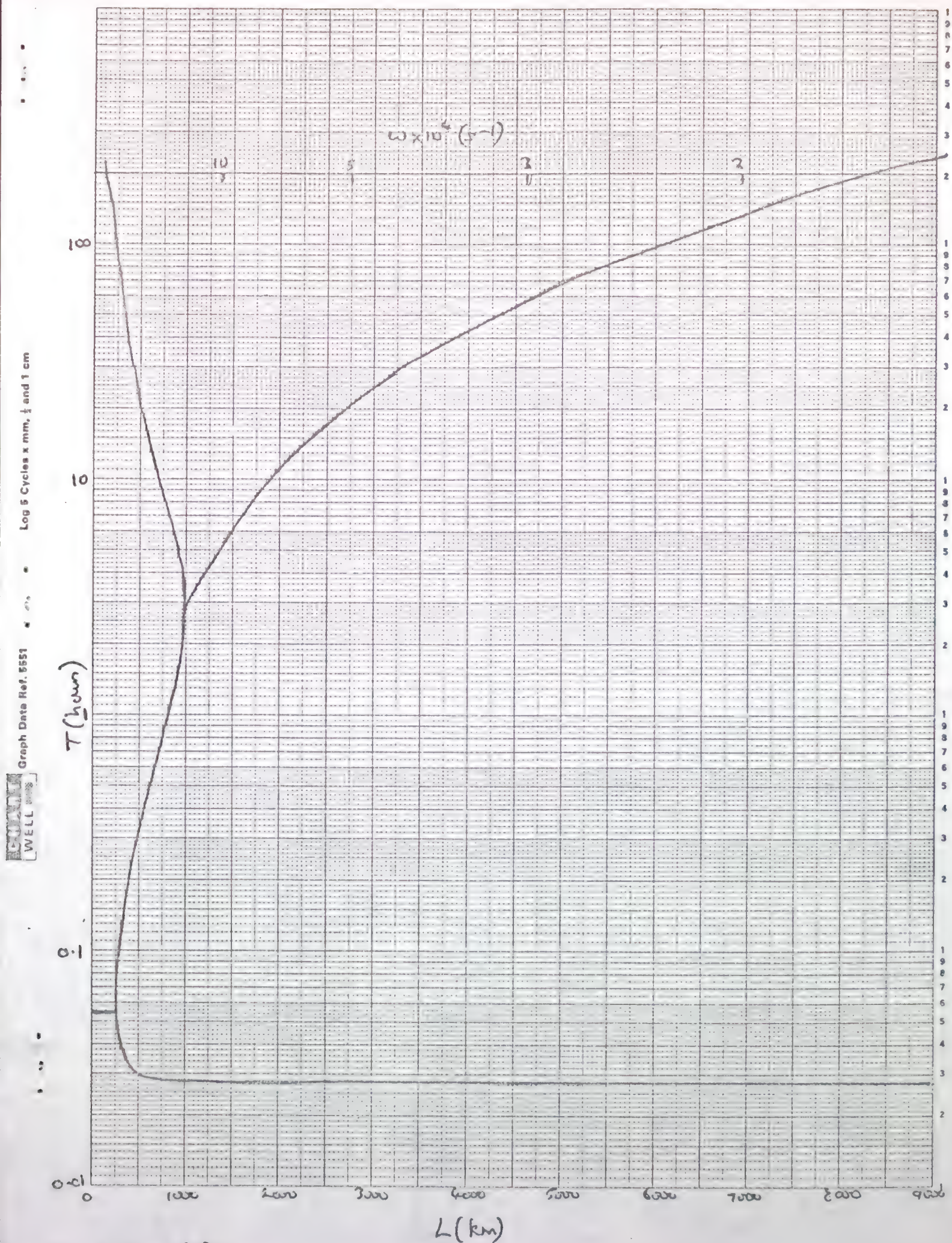

$\omega \times 10^4 (s^{-1})$
10
5
3
2
100
10
0·1
0·01
T (hrs)
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
L (km)
Log 5 Cycles x mm, ½ and 1 cm
Graph Data Ref. 5551
WELL

Figure 2.2a

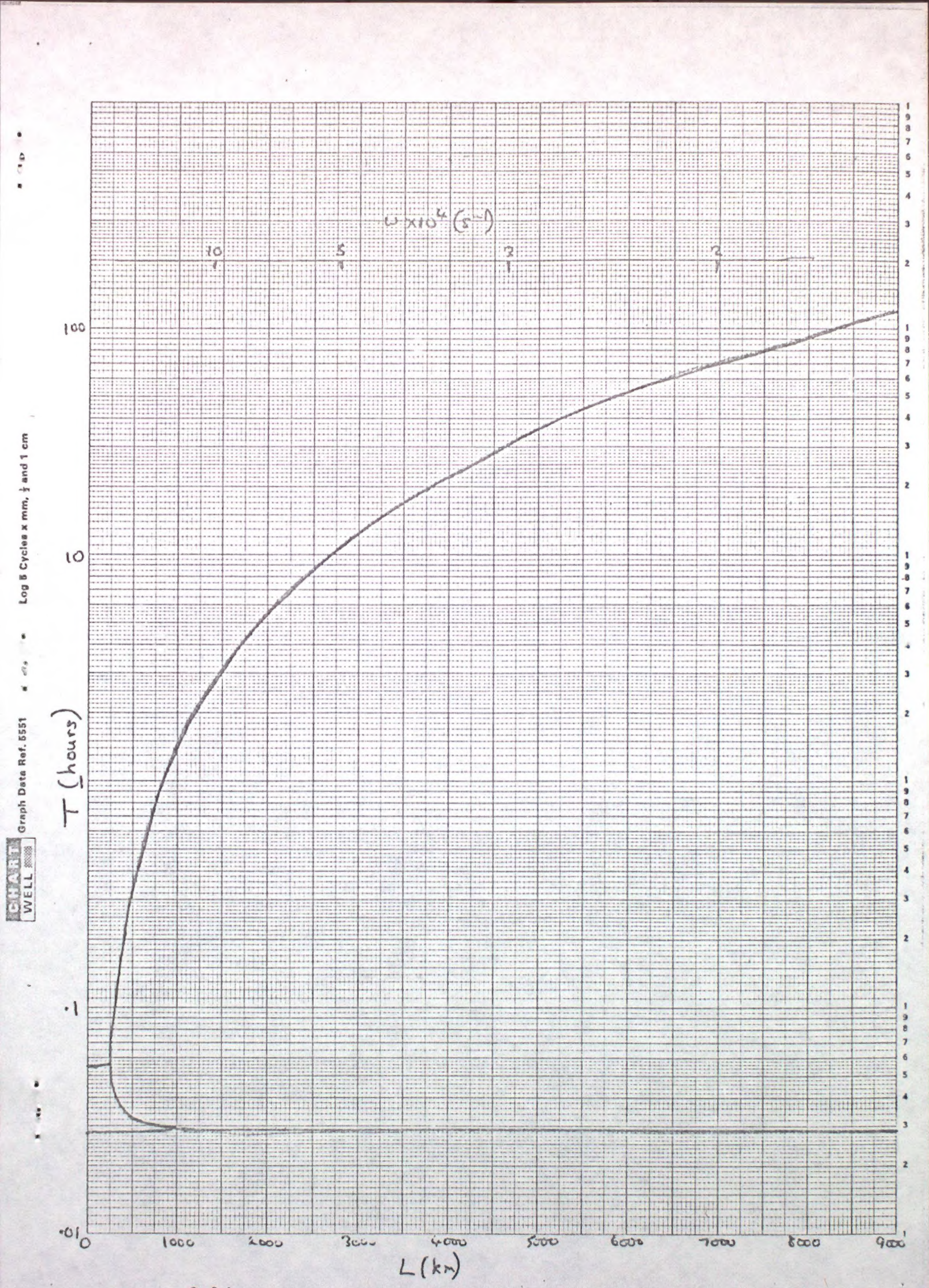
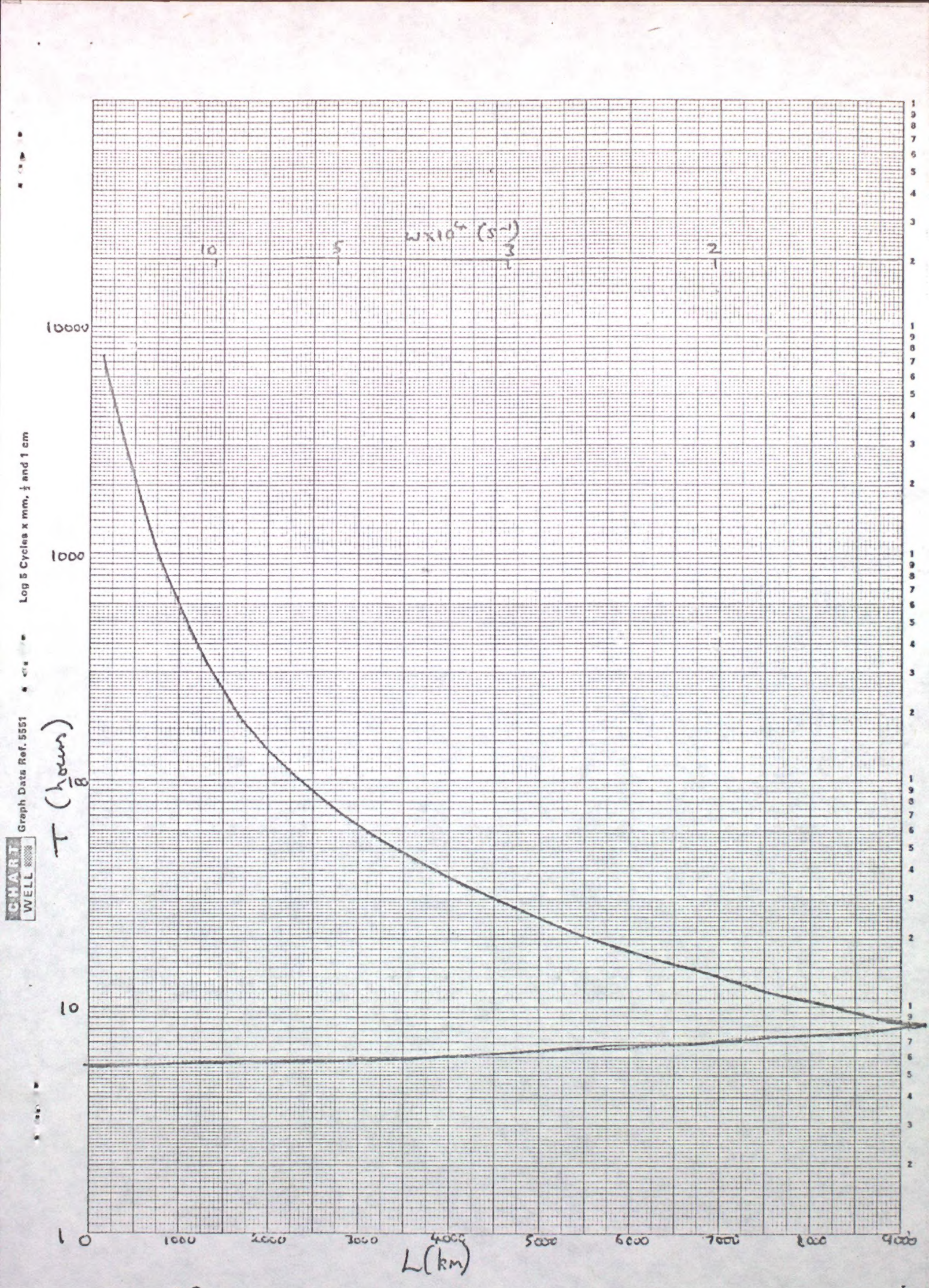
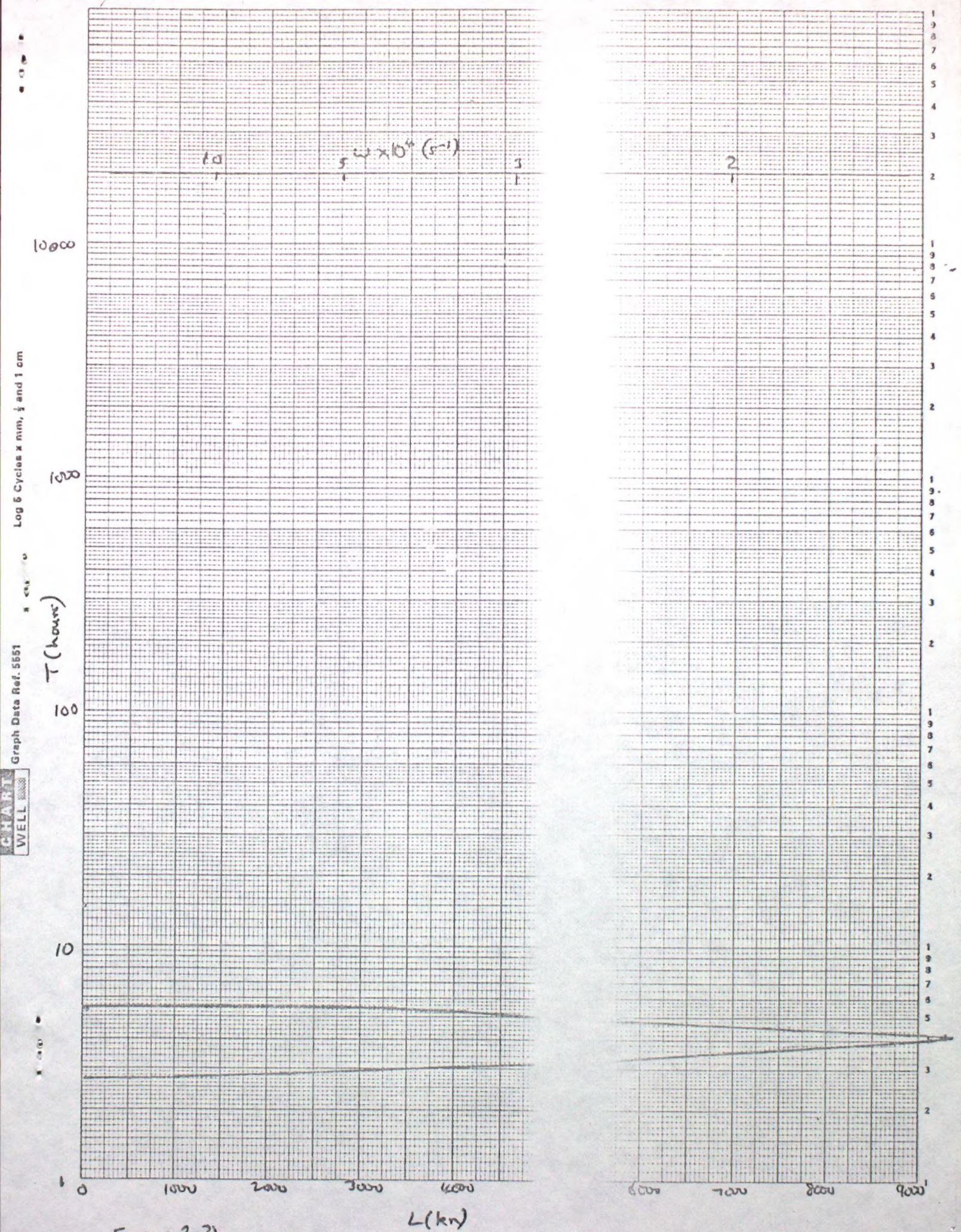

ω x10⁴ (s⁻¹)
10
5
3
2
100
10
1
·1
·01
T (hours)
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
L (km)
Log 5 Cycles x mm, ½ and 1 cm
Graph Data Ref. 5551
CHART WELL

Figure 2.2b



ω×10⁴ (s⁻¹)
10
5
3
2
10000
1000
100
10
1
T (hours)
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
L(km)
Log 5 Cycles x mm, ½ and 1 cm
Graph Data Ref. 5551
CHART WELL

Figure 2.3a



ω × 10⁴ (s⁻¹)
10
5
3
2
10000
1000
100
10
1
T (hours)
0
1000
2000
3000
4000
6000
7000
8000
9000
L (km)
Log 5 Cycles x mm, ½ and 1 cm
Graph Data Ref. 5551
CHART WELL

Figure 2.3b

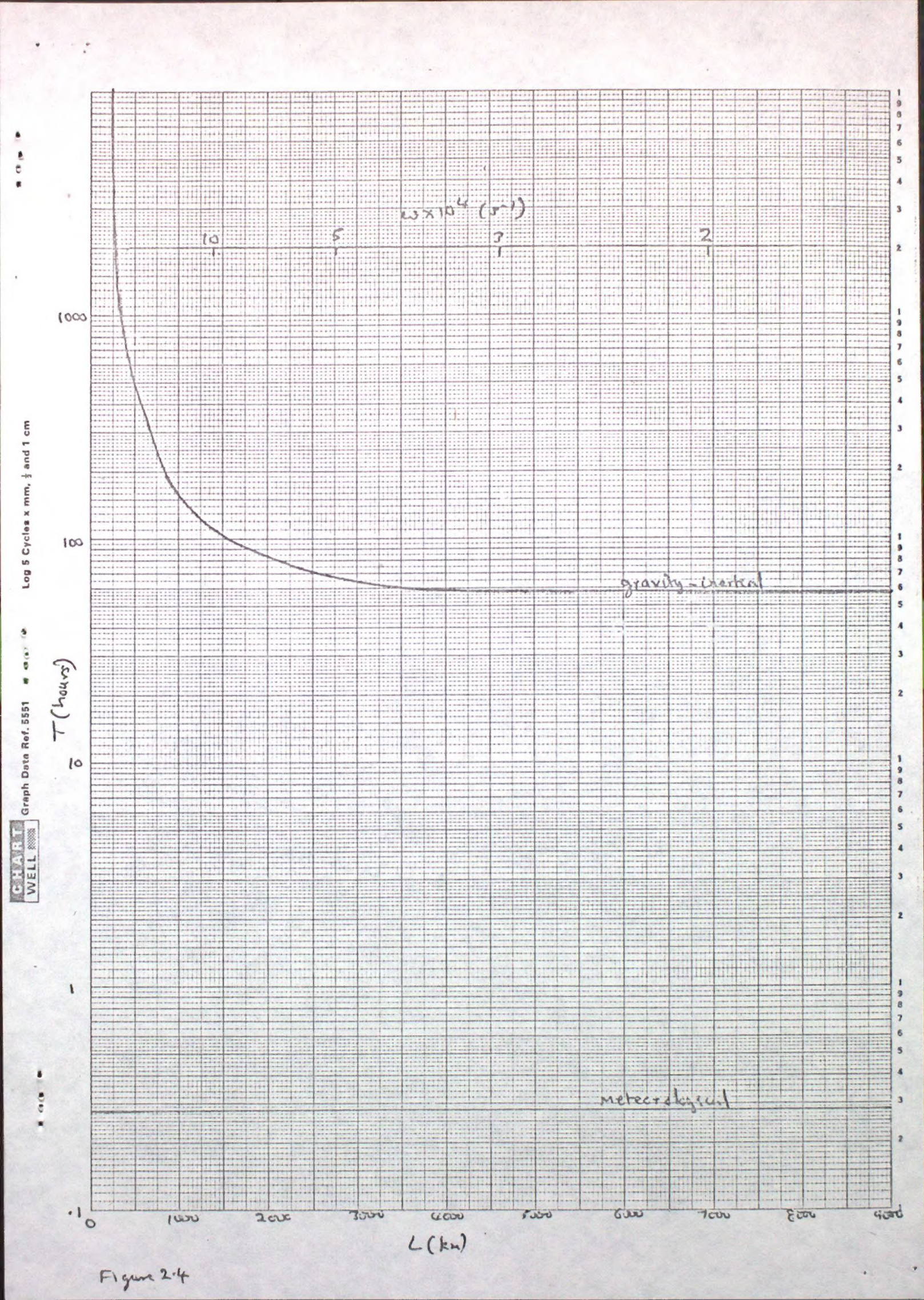

2 x 10^-4 (s^-1)
10
5
3
2
1000
100
10
1
.1
T (hours)
gravity - inertial
meteorological
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
L (km)
Log 5 Cycles x mm, ½ and 1 cm
Graph Data Ref. 5551
CHART WELL

Figure 2.4